Correct Aggregation Operations with Algorithms

Construction of correct algorithms from other correct algorithms using aggregation operations

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What is about

Consider models of algorithms and learning methods which make possible to build *collections* of *correct algorithms*.

Correct algorithm produces correct output for all learning samples and we call them *basic* correct algorithms

Question

How to build new correct algorithms from other correct algorithms that could extend capabilities of basic correct algorithms?

Some kinds of *aggregation operations* can be used for building new correct algorithms from other correct algorithms

Initial definitions

$\mathbf{X}$ – space of input informations about objects

$\mathbf{Y}$ – set of possible answers to given question about objects

$y\colon \mathbf{X} \to \mathbf{Y}$ – mapping from input informations to answers

$\mathfrak{a}\colon \mathbf{X} \to \mathbf{Y}$ – algorithm ($\mathfrak{a} \in \mathbb{A}$), which *approximates* y

$\mathbb{A}$ – space of *basic* algorithms

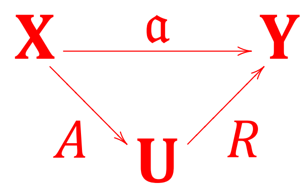
$\widetilde{\mathbf{X}}$ – finite subset of $\mathbf{X}$ (*samples*)

$\mathfrak{a}$ is *correct* on $\widetilde{\mathbf{X}}$ if it evaluates *valid answers* on $\widetilde{\mathbf{X}}$

F is *correct operation* with algorithms from $\mathbb{A}$ on $\widetilde{\mathbf{X}}$ if for any tuple $\{\mathfrak{a}_1, \dots, \mathfrak{a}_m\}$ of correct algorithms on $\widetilde{\mathbf{X}}$ the new algorithm $\mathfrak{a} = \mathrm{F}\{\mathfrak{a}_1, \dots, \mathfrak{a}_m\}$ is correct on $\widetilde{\mathbf{X}}$

Quality functions of answers

Consider algorithms that represented in the form $\mathfrak{a} = R \circ A$:



A – *estimating operator*, R – *decision rule*

$\mathbf{U} \subseteq \mathbb{R}^q$ – *space of estimations*

Quality function of answer $Q(\mathbf{u}, \tilde{y})$ evaluates quality of answer $y = \mathfrak{a}(\tilde{\mathbf{x}})$ on the base of estimation $\mathbf{u} = A(\tilde{\mathbf{x}})$ and correct answer $\tilde{y}$

Subset $\mathbf{Q}_{\tilde{y}} \subset \mathrm{image}\, Q$ associates with answers $y = \mathfrak{a}(\tilde{\mathbf{x}})$ that accepted as correct as $\tilde{y}$

Answer $y = \mathfrak{a}(\tilde{\mathbf{x}})$ is *correct* if $Q(\mathbf{u} \mid \tilde{y}) \in \mathbf{Q}_{\tilde{y}}$.

Monotone/unimodal functions

$f\colon \mathbf{U} \to \mathbb{R}$ – function of many arguments

f is *monotone* if $\forall y \in \mathbb{R}$: $f^{-1}(y)$ is connected set

Denote $D_f(y) = \{\mathbf{u} : f(\mathbf{u}) \leqslant y\}$

f is *monotone increasing* if $\forall y \in \mathbb{R}$:

$$y_1 \leqslant y_2 \Longrightarrow D_f(y_1) \subseteq D_f(y_2)$$

f is *monotone decreasing* if $\forall y \in \mathbb{R}$:

$$y_1 \geqslant y_2 \Longrightarrow D_f(y_1) \subseteq D_f(y_2)$$

f is *unimodal* if the set $\{\mathbf{u} : \mathbf{u}$ is local min/max of $f\}$ is connected

Loss functions

Loss function is mapping $Q: \mathbf{U} \times \mathbf{Y} \to \mathbb{R}_+$ if
for any $\tilde{y}$: $Q_{\tilde{y}}(\mathbf{u}) = Q(\mathbf{u}, \tilde{y})$ is:

- *monotone*,
- *increasing*,
- *unimodal* (min).

- $Q_{\tilde{y}} = \min_{\mathbf{u} \in \mathbf{U}} Q(\mathbf{u}, \tilde{y})$ – minimal losses for correct answers
- $\mathbf{U}_{\tilde{y}} = \{\mathbf{u}: Q(\mathbf{u}, \tilde{y}) = Q_{\tilde{y}}\}$ – set of estimates that associates with correct answer $\tilde{y}$

Margin functions

Margin function is mapping $Q\colon \mathbf{U} \times \mathbf{Y} \to \bar{\mathbb{R}}$ if for any $\tilde{y}$: $Q_{\tilde{y}}(\mathbf{u}) = Q(\mathbf{u}, \tilde{y})$ is:

- *monotone*,
- *unimodal* (max).

- $\mathbf{U}_{\tilde{y}} = \{\mathbf{u}\colon Q(\mathbf{u}, \tilde{y}) > 0\}$ – set of estimates that associates with correct answer
- $\bar{\mathbf{U}}_{\tilde{y}} = \{\mathbf{u}\colon Q(\mathbf{u}, \tilde{y}) < 0\}$ – set of estimates that associates with incorrect answer
- $\delta\mathbf{U}_{\tilde{y}} = \{\mathbf{u}\colon Q(\mathbf{u}, \tilde{y}) = 0\}$ – set of estimates when definite answer isn't possible

Regression problems ($\mathbf{Y} \subset \mathbb{R}$)

$Q(\mathfrak{a} \mid \mathbf{x}) = \ell(\mathfrak{a}(\tilde{\mathbf{x}}), \tilde{y})$ – *loss function*, $\mathbf{V}_Q = [0, \varepsilon]$

Definition

Function $\ell(y, \tilde{y})$ is a loss function if:

1. $\ell(\tilde{y}, \tilde{y}) = \inf_y \ell(y, \tilde{y})$
2. $y_1 \leqslant y_2 \leqslant \tilde{y} \Rightarrow \ell(y_1, \tilde{y}) \geqslant \ell(y_2, \tilde{y}) \geqslant \ell(\tilde{y}, \tilde{y})$
3. $\tilde{y} \leqslant y_1 \leqslant y_2 \Rightarrow \ell(\tilde{y}, \tilde{y}) \leqslant \ell(y_1, \tilde{y}) \leqslant \ell(y_2, \tilde{y})$

Definitions (correct answer)

- answer $y = \mathfrak{a}(\tilde{\mathbf{x}})$ is as correct as $\tilde{y}$ if $\ell(y, \tilde{y}) \in \mathbf{V}_Q$.
- answer $y = \mathfrak{a}(\tilde{\mathbf{x}})$ is correct if $y \in \mathbf{U}_{\tilde{y}} = \{y \in \mathbf{Y} \colon \ell(y, \tilde{y}) \in \mathbf{V}_Q\}$.

Examples: loss functions

$$\langle y-\tilde{y}\rangle_\varepsilon = \left(\frac{|y-\tilde{y}|}{1+|\tilde{y}|}-\varepsilon\right)_+, E_+ = [E \geqslant 0]\cdot E, \varepsilon > 0$$

Example (symmetrical)

$$\ell(y,\tilde{y}) = \langle y-\tilde{y}\rangle_\varepsilon^\lambda, \mathbf{V}_Q = [0,\varepsilon)$$

Example (nonsymmetrical)

$$\ell(y,\tilde{y}) = \begin{cases} \alpha\langle y-\tilde{y}\rangle_\varepsilon^\lambda & \text{if } y > \tilde{y}+\varepsilon \\ 0, & \text{if } |y-\tilde{y}| \leqslant \varepsilon, \mathbf{V}_Q = [0,\varepsilon), \alpha,\beta > 0 \\ \beta\langle y-\tilde{y}\rangle_\varepsilon^\lambda & \text{if } y < \tilde{y}-\varepsilon \end{cases}$$

Classification problems ($\mathbf{Y}$ is discrete)

Algorithms for classification are compositions:
$\mathfrak{a}(\mathbf{x}) = R \circ A(\mathbf{x})$.

Quality functions:

- $Q(\mathfrak{a} \mid \mathbf{x}) = \mu(\mathbf{u}, \tilde{y})$, $\mu\colon \mathbf{U} \times \mathbf{Y} \to \bar{\mathbb{R}}$ is *margin function* and $\mathbf{V}_Q = [\delta, \infty]$
- $Q(\mathfrak{a} \mid \mathbf{x}) = \ell(\mathbf{u}, \tilde{y})$, $\ell\colon \mathbf{U} \times \mathbf{Y} \to \mathbb{R}_+$ is *loss function* and $\mathbf{V}_Q = [0, \varepsilon]$

Definitions (correct estimate)

- estimation $\mathbf{u} = A(\mathbf{x})$ is correct if $\mu(\mathbf{u}, \tilde{y}) \in \mathbf{V}_Q$
- estimation $\mathbf{u} = A(\mathbf{x})$ is correct if $\mathbf{u} \in \mathbf{U}_{\tilde{y}} = \{\mathbf{u}\colon \mu(\mathbf{u}, \tilde{y}) \in \mathbf{V}_Q\}$

Examples: classification

Example (2-class classification)

$\mathbf{Y} = \{-1, 0, +1\}$, $R(u) = \operatorname{sign} u$, $\mu(u, \tilde{y}) = uy$

$\mathbf{V}_Q = [\delta, \infty]$, $\mathbf{U}_{\tilde{y}} = \{u\colon u\tilde{y} > \delta\}$.

Example (q-class classification)

$\mathbf{Y} = \{0, 1, \dots, q\}$, $\mathbf{A} = (A_1, \dots, A_m)$, $u_j = A_j(\mathbf{x})$

$$R(\mathbf{u}) = \begin{cases} y^\star = \arg\max_{y \in \{1,\dots,m\}} \{u_y\}, & \text{if } \mu(\mathbf{u}, \tilde{y}) \geqslant \varepsilon \\ \text{fail}, & \text{if } \mu(\mathbf{u}, \tilde{y}) \in (-\varepsilon, \varepsilon) \\ 0, & \text{if } \mu(\mathbf{u}, \tilde{y}) \leqslant -\varepsilon \end{cases}$$

$\ell(\mathbf{u}, \tilde{y}) = e^{-\mu(\mathbf{u}, \tilde{y})}$, $\qquad \mu(\mathbf{u}, \tilde{y}) = u_{\tilde{y}} - \max_{j = \{1,\dots,q\} \setminus \{\tilde{y}\}} \{u_j\}$

$\mathbf{V}_Q = (\varepsilon, \infty)$, $\mathbf{U}_{\tilde{y}} = \{\mathbf{u}\colon u_{\tilde{y}} > u_y \text{ for all } y \neq \tilde{y}\}$

Aggregation and mean functions

Definition (aggregation function)

M – *aggregation function* on $\mathbf{Y} \subseteq \mathbb{R}$, i.e.

- for any $m \in \mathbb{N}$ and any tuple $\{y_1, \dots, y_m\}$: $\mathrm{M}\{y_1, \dots, y_m\} \in \mathbf{Y}$
- for any $m \in \mathbb{N}$ and any pair of tuples $\{y_1', \dots, y_m'\} \leqslant \{y_1'', \dots, y_m''\}$: $\mathrm{M}\{y_1', \dots, y_m'\} \leqslant \mathrm{M}\{y_1'', \dots, y_m''\}$

Definition (mean function)

M – *mean function* if

$$\min\{y_1, \dots, y_m\} \leqslant \mathrm{M}\{y_1, \dots, y_m\} \leqslant \max\{y_1, \dots, y_m\}$$

Definition (idempotent function)

M – *idempotent function* if

$$\mathrm{M}\{y, \dots, y\} = y$$

Point-wise correct operations

Definition (point-wise correct algorithm)

$\mathfrak{a}$ is *correct algorithm* on $\tilde{\mathbf{X}}$ if $\mathfrak{a}(\tilde{\mathbf{x}})$ is correct for all $\tilde{\mathbf{x}} \in \tilde{\mathbf{X}}$

Condition (point-wise correct operation)

M is *correct operation* if for any tuple $\{y_1, \ldots, y_m\} \subset \mathbf{U}_{\tilde{y}}$:

$$\mathrm{M}\{y_1, \ldots, y_m\} \in \mathbf{U}_{\tilde{y}}$$

Criterion (mean function – correct point-wise operation)

If M is idempotent mean function, then M is correct operation.

Generalized Kolmogorov's mean

Definition (generalized Kolmogorov's mean)

$$\mathrm{M}_g\{u_1,\dots,u_m\} = g^{-1}\big(w_1 g(u_1) + \cdots + w_m g(u_m)\big),$$

$w_1,\dots,w_m \geqslant 0$ and $w_1 + \cdots + w_m = 1$

$g : \mathbf{U} \to \mathbb{R}$ or $g : \mathbf{U} \to \mathbb{R}_+$ or $g : \mathbf{U} \to [0,1]$ – inversible function

Examples (weighted Kolmogorov's mean)

- $\mathrm{M}_g\{u_1,\dots,u_m\} = (\sum w_j u_j^{\langle p\rangle})^{\langle 1/p\rangle}$ $\qquad g(s) = u^{\langle p\rangle}$
- $\mathrm{M}_g\{u_1,\dots,u_m\} = \frac{1}{p}\ln(\sum w_j e^{p u_j})$ $\qquad g(u) = e^{pu}$
- $\mathrm{M}_g\{u_1,\dots,u_m\} = \prod u_j^{w_j}$ $\qquad g(u) = \ln u$

Mean aggregation operators

Let $\preccurlyeq$ – partial order on $\mathbf{U}$

Definition

$\mathsf{M} = (\mathsf{M}_1, \dots, \mathsf{M}_m)$ is *aggregation operator* on $\mathbf{U}$:

- for each m and any tuple $\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \subset \mathbf{U}$:
  $\mathsf{M}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \in \mathbf{U}$;
- for each m and any pairs of tuples
  $\{\mathbf{u}'_1, \dots, \mathbf{u}'_m\} \preccurlyeq \{\mathbf{u}''_1, \dots, \mathbf{u}''_m\}$:
  $\mathsf{M}\{\mathbf{u}'_1, \dots, \mathbf{u}'_m\} \preccurlyeq \mathsf{M}\{\mathbf{u}''_1, \dots, \mathbf{u}''_m\}$

Definition

M is *mean aggregation operator* if

$$\inf\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \preccurlyeq \mathsf{M}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \preccurlyeq \sup\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$$

M is point-wise correct operation if $\mathsf{M}\{\mathbf{U}_{\tilde{y}}, \dots, \mathbf{U}_{\tilde{y}}\} \subseteq \mathbf{U}_{\tilde{y}}$.

Multivariate Kolmogorov's mean

If $\mathbf{M}$ is idempotent operation, i.e. $\mathbf{M}\{\mathbf{u},\ldots,\mathbf{u}\}=\mathbf{u}$, then $\mathbf{M}$ is correct point-wise operation.

Multivariate weighted Kolmogorov's mean:

$$\mathbf{M}_{\boldsymbol{g}}\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}=\boldsymbol{g}^{-1}\Big(\sum \mathbf{w}_j\boldsymbol{g}(\mathbf{u}_j)\Big),$$

where $\mathbf{w}_1+\cdots+\mathbf{w}_m=1$, $\boldsymbol{g}\colon \mathbf{U}\to\mathbb{R}^q$ is inversible continuous mapping.

Fact

If $\boldsymbol{g}(\mathbf{U}_{\tilde{y}})$ is a convex set then $\mathbf{M}_{\boldsymbol{g}}\{\mathbf{U}_{\tilde{y}},\ldots,\mathbf{U}_{\tilde{y}}\}\subseteq\mathbf{U}_{\tilde{y}}$, i.e. multivariate weighted Kolmogorov's mean $\mathbf{M}_{\boldsymbol{g}}$ is correct point-wise operation.

Aggregationally correct algorithm

M – aggregation function on image $\mathcal{Q}$.

Aggregation quality functional

$$\mathcal{Q}(\mathfrak{a} \mid \widetilde{\mathbf{X}}) = \mathrm{M}\{\mathcal{Q}(\mathfrak{a} \mid \tilde{\mathbf{x}}) : \tilde{\mathbf{x}} \in \widetilde{\mathbf{X}}\}$$

evaluates the quality of algorithm on whole set $\widetilde{\mathbf{X}}$ on the base of quality of all answers on $\widetilde{\mathbf{X}}$.

Let $\mathbf{V}_{\mathcal{Q}}$ is subset of values of $\mathcal{Q}$, which corresponds correct algorithms on $\widetilde{\mathbf{X}}$.

Definition (Aggregationally correct algorithm)

Algorithm $\mathfrak{a}$ is *aggregationally correct* on $\widetilde{\mathbf{X}}$ if $\mathcal{Q}(\mathfrak{a} \mid \widetilde{\mathbf{X}}) \in \mathbf{V}_{\mathcal{Q}}$.

Linear operations

M – arithmetic weigted mean:

$$\mathrm{M}\{z_1, \ldots, z_N\} = \sum_{k=1}^{N} w_k z_k$$

F – linear operation:

$$\mathrm{F}\{u_1, \ldots, u_m\} = \alpha_1 u_1 + \cdots + \alpha_m u_m,$$

$\alpha_1, \ldots, \alpha_m \geqslant 0, \alpha_1 + \cdots + \alpha_m = 1.$

Let

1. $Q_{\tilde{y}}(\mathbf{u})$ – *convex for any* $\tilde{y} \in \mathbf{Y}$ *;*
2. $\mathbf{V}_Q$ *is convex;*
3. *if* $Q' \in \mathbf{V}_Q$ *u* $Q'' \leqslant Q'$, *mo* $Q'' \in \mathbf{V}_Q$.

Then linear operation is aggreagtionally correct operation with algorithms with respect to Q.

F/G-convex

$f\colon \mathbf{U} \to \mathbf{R}$

F – idempotent aggregation function on $\mathbf{U} \subseteq \mathbb{R}$

G – idempotent aggregation function on $\mathbf{R} \subseteq \mathbb{R}$

Definition

$f(\mathbf{u})$ is F/G-convex, if

$$f(\mathrm{F}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}) \leqslant \mathrm{G}\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_m)\}.$$

M – aggregation function on $\mathbf{R}$

Example

$$f(\alpha_1\mathbf{u}_1 + \cdots + \alpha_m\mathbf{u}_m) \leqslant \alpha_1 f(\mathbf{u}_1) + \cdots + \alpha_m f(\mathbf{u}_m)$$

$$f(\alpha_1\mathbf{u}_1 + \cdots + \alpha_m\mathbf{u}_m) \leqslant \max\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_m)\}$$

Domination of aggregation functions

Definition

G is dominated over M if

$$\mathrm{M}\{\mathrm{G}\{u_{11},\dots,u_{1m}\},\dots,\mathrm{G}\{u_{N1},\dots,u_{Nm}\}\} \leqslant \mathrm{G}\{\mathrm{M}\{u_{11},\dots,u_{N1}\},\dots,\mathrm{M}\{u_{1m},\dots,u_{Nm}\}\}$$

If $\forall N \in \mathbb{N}$ *function* $H(u_1,\dots,u_N) = \mathrm{M}\{u_1,\dots,u_N\}$ *is convex, then*

$$\mathrm{M}\Big\{\sum_j \alpha_j u_{1j},\dots,\sum_j \alpha_j u_{Nj}\Big\} \leqslant \sum_j \alpha_j \mathrm{M}\{u_{1j},\dots,u_{Nj}\},$$

i.e. linear weighted mean dominated over M.

Aggregationally correct operations

Theorem

Let

1. $Q_{\tilde{y}}(\mathbf{u})$ – F/G-convex for any $\tilde{y} \in \mathbf{Y}$;
2. G is dominated over M;
3. $\mathbf{V}_Q$ is closed within G;
4. if $Q' \in \mathbf{V}_Q$ и $Q'' \leqslant Q'$, то $Q'' \in \mathbf{V}_Q$.

Then F – aggreagtionally correct operation with algorithms with respect to Q.

From domination to convexity

G – aggregation operation on $\mathbb{R}$

F – aggregation operation on $\mathbb{R}^m$:

$$\mathrm{F}\{\mathbf{u}_1, \dots, \mathbf{u}_N\} = (\mathrm{G}\{u_{11}, \dots, u_{1m}\}, \dots, \mathrm{G}\{u_{N1}, \dots, u_{Nm}\})$$

H – aggregation operation on image of M

Definition

M – G/H-convex if

$$\mathrm{M}\{\mathrm{G}\{u_{11}, \dots, u_{1m}\}, \dots, \mathrm{G}\{u_{N1}, \dots, u_{Nm}\}\} \leqslant \mathrm{H}\{\mathrm{M}\{u_{11}, \dots, u_{N1}\}, \dots, \mathrm{M}\{u_{1m}, \dots, u_{Nm}\}\}.$$

Aggregationally correct operations

Theorem

Let

1. $Q_{\tilde{y}}(\mathbf{u})$ – F/G-convex;
2. M – G/H-convex;
3. $\mathbf{V}_Q$ is closed within H;
4. if $Q' \in \mathbf{V}_Q$ and $Q'' \leqslant Q'$ then $Q'' \in \mathbf{V}_Q$.

Then F – aggreagtionally correct operation with algorithms with respect to Q.

Thank you!